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Embedding of Dynamical Symmetry Groups $U(1, 1)$ and $U(2)$ of a Free Particle on AdS_2 and S^2 into Parasupersymmetry Algebra

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Using two different types of the laddering equations realized simultaneously by the associated Gegenbauer functions, we show that all quantum states corresponding to the motion of a free particle on AdS_2 and S^2 are splitted into infinite direct sums of infinite- and finite-dimensional Hilbert subspaces which represent Lie algebras $u(1, 1)$ and $u(2)$ with infinite- and finite-fold degeneracies, respectively. In addition, it is shown that the representation bases of Lie algebras with rank 1, i.e., $gl(2, C)$, realize the representation of nonunitary parasupersymmetry algebra of arbitrary order. The realization of the representation of parasupersymmetry algebra by the Hilbert subspaces which describe the motion of a free particle on AdS_2 and S^2 with the dynamical symmetry groups $U(1, 1)$ and $U(2)$ are concluded as well.

KEY WORDS: dynamical symmetry; parasupersymmetry algebra; Lie algebra; shape invariance.

1. INTRODUCTION

Initially the factorization method has been suggested by Darboux. The generalization of the method was provided by Schrödinger (1940, 1941a,b) in the context of quantum mechanics. Infeld and Hull (1951) in their review article have shown a large variety of second-order differential equations with boundary conditions set in six different types of factorization (the details of the connection of those to each other have been considered in the recent articles (Del Sol Mesa and Quesne, 2000, 2002)). On other hand, serious efforts of physicists for describing unified of basic interactions in nature have brought them up to the result in which supersymmetry is one of the requirement ingredient for this approach. The idea of supersymmetry in context of quantum mechanics were first studied by Nicolai

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(1976) and Witten (1981) and later by Cooper and Freedman (1983). At this time, Gendenshtein forwarded the concept of shape invariance in the framework of supersymmetric quantum mechanics (1983). From his idea, the supersymmetric partner potential corresponding to an original potential has a same spatial functionality with it, so that the suitable parameters just have shifted. He has shown that for any shape-invariant potential, quantum states of consecutive spectral can be calculated by algebraic method. Later on, a list of shape-invariant potentials derived and it has shown that their eigenstates and scattering matrix can be calculated by the algebraic method (Cooper *et al.*, 1988; Dabrowska *et al.*, 1988; Dutt *et al.*, 1986; Khare and Sukhatme, 1988). Frequently, it was found out that factorization method of Infeld and Hull is related to the shape-invariant potentials formalism in the context of supersymmetric quantum mechanics (Infeld and Hull, 1951). Till now, several studies on the one-dimensional shape invariant potentials have been accomplished (Adrianov *et al.*, 1984; Balantekin, 1998; Balantekin *et al.*, 1999; Carinena and Ramos, 2000a,b; Cooper *et al.*, 1995; Chuan, 1998; Das and Huang, 1990; Dutt *et al.*, 1988; Fukui and Aizawa, 1993; Gendenshtein and Krive, 1985; Haymaker and Rau, 1986; Salamoson and van Holten, 1982; Sukumar, 1985). Recently, by application of master function theory and also main and secondary quantum numbers, i.e., n and m, one-dimensional shape invariant models have been classified in two different classes (Fakhri, 2003). The first and second classes are shape invariant with respect to n and m, respectively. The recent subject is one of the most powerful results for one-dimensional models in context of supersymmetry, because shape invariance not only represents $\mathcal{N} = 2$ supersymmetry algebra (Jafarizadeh and Fakhri, 1977) but also parasupersymmetry algebra of arbitrary order p (Jafarizadeh and Fakhri, 1998).

The theory of parasupersymmetric quantum mechanics is about parafermion and paraboson particles, fermions and bosons obey ordinary statistics and parabosons and parafermions are assumed to obey an intermediate kind of statistics, called parastatistics (Green, 1953; Ohnuki and Kamefuchi, 1982; Volkov, 1960). For symmetry between boson and parafermion in case $p = 2$ parastatistics, first time Rubakov and Spiridonov suggested parasupersymmetry algebra of order $p = 2$ (Rubakov and Spiridonov, 1988) which was extended by Khare for arbitrary order p (Khare, 1992, 1993). There are useful discussions about parasupersymmetry algebra of order $p = 2$ in Mostafazadeh (1996, 1997). In addition to representation of Khare–Rubakov–Spiridonov parasupersymmetry algebra (Jafarizadeh and Fakhri, 1998), shape invariance of one-dimensional solvable models leads to derivation of two-dimensional quantum mechanical models including dynamical symmetry groups, which are also embedded into parasupersymmetry algebra of arbitrary order p.

In this paper, introducing the associated Gegenbauer functions in terms of two nonnegative integers n and m with $n \geq 0$, $0 \leq m \leq n$, we factorize their corresponding differential equation as shape invariance equations with respect to the parameters n and m. Then, we determine the normalization coefficients and consequently their norms with respect to an appropriate inner product, so that the associated Gegenbauer functions satisfy the laddering relations with respect to n and m, simultaneously. On the basis of representation of the laddering equations with respect to $n\backslash m$, we construct the generators of Lie algebra $u(1,1)\backslash u(2)$ together with infinite-$\backslash(n+1)$-dimensional representation space $\mathcal{H}_m\backslash\mathcal{H}_n$. It is also shown that the Casimir operator of the Lie algebra $u(1,1)\backslash u(2)$ is the Hamiltonian operator corresponding to the motion of a free particle on $AdS_2\backslash S^2$ with infinite-$\backslash(n+1)$-fold degeneracy in $\mathcal{H}_m\backslash\mathcal{H}_n$. In this manner, all permissible quantum states of a free particle on $AdS_2\backslash S^2$ are splitted to an infinite direct sum of the Hilbert subspace $\mathcal{H}_m\backslash\mathcal{H}_n$. Using laddering relations with respect to $m\backslash n$ we introduce the differential operators of order 1 such that the Hilbert subspaces $\mathcal{H}_m\backslash\mathcal{H}_n$ map to each other. Meanwhile, we show that $gl(2,C)$, i.e., complexification of Lie algebras with rank 1, represents Kahre–Rubakov–Spiridonov nonunitary parasupersymmetry algebra of arbitrary order p. For this, we use Cartan bases K_+, K_- and K_3 of Lie algebra $gl(2,C)$ to make parafermionic and bosonic operators, and also use the representation bases of this algebra to make parastates. We show that the components of bosonic operator participate in $p+1$ eigenvalue equations with one common eigenvalue but with the $p+1$ successive representation bases of Lie algebra $gl(2,C)$. Each of real forms or Lie subalgebra $gl(2,C)$ which are included the generators K_+, K_- and K_3 can represent the Kahre–Rubakov–Spiridonov nonunitary parasupersymmetry algebra. As application, we can also allocate our discussion to realize the representation of parasupersymmetry algebra by the bases of Hilbert subspaces $\mathcal{H}_m\backslash\mathcal{H}_n$, corresponding to the motion of a free particle on $AdS_2\backslash S^2$ with dynamical symmetry group $U(1,1)\backslash U(2)$.

2. SHAPE INVARIANCE AND LADDERING EQUATIONS WITH RESPECT TO *n* AND *m* FOR THE ASSOCIATED GEGENBAUER FUNCTIONS

2.1. The Gegenbauer Polynomials $P_n^{(\lambda)}(x)$

For given real parameter $\lambda > -1$ and the real variable $-1 < x < +1$, the differential equation of Gegenbauer polynomials $P_n^{(\lambda)}(x)$ is known as

$$(1-x^2)P_n''^{(\lambda)}(x) - 2(\lambda+1)xP_n'^{(\lambda)}(x) + n(2\lambda+n+1)P_n^{(\lambda)}(x) = 0. \tag{1}$$

One may deduce that the Gegenbauer polynomials $P_n^{(\lambda)}(x)$ have the Rodrigues representation as follows (Nikiforov and Uvarov, 1988)

$$P_n^{(\lambda)}(x) = \frac{a_n(\lambda)}{(1-x^2)^\lambda}\left(\frac{d}{dx}\right)^n((1-x^2)^{\lambda+n}), \tag{2}$$

where $a_n(\lambda)$ is a real normalization coefficient. Using the Rodrigues representation (2), one can calculate the coefficient of the highest power of x for the Gegenbauer polynomials of degree n as

$$P_n^{(\lambda)}(x) = a_n(\lambda)(-1)^n \frac{\Gamma(2\lambda + 2n + 1)}{\Gamma(2\lambda + n + 1)} x^n + O(x^{n-1}). \tag{3}$$

It is easy to show that the polynomials $P_n^{(\lambda)}(x)$ with different n's, respect to the inner product with weight function $(1 - x^2)^\lambda$ in the interval $x \in (-1, +1)$ are orthogonal. Hence, by choosing the normalization coefficient $a_n(\lambda)$ as

$$a_n(\lambda) = \sqrt{\frac{(2\lambda + 2n + 1)\Gamma(2\lambda + n + 1)}{2^{2\lambda+2n+1}\Gamma(n + 1)}} \frac{h_n(\lambda)}{\Gamma(\lambda + n + 1)}, \tag{4}$$

we can see that

$$\int_{-1}^{+1} P_n^{(\lambda)}(x) P_{n'}^{(\lambda)}(x)(1 - x^2)^\lambda \, dx = \delta_{nn'} h_n^2(\lambda) \qquad n, n' \geq 0, \tag{5}$$

where $h_n(\lambda)$ is norm of the Gegenbauer polynomial $P_n^{(\lambda)}(x)$.

2.2. The Associated Gegenbauer Functions $P_{nm}^{(\lambda)}(x)$

By taking m times derivative from the Gegenbauer polynomials differential equation (1) and change of function by multiplicity $(1 - x^2)^{\frac{m}{2}}$, we obtain the differential equation of the associated Gegenbauer functions $P_{n,m}^{(\lambda)}(x)$ as

$$(1 - x^2) P_{n,m}^{\prime\prime(\lambda)}(x) - 2(\lambda + 1) x P_{n,m}^{\prime(\lambda)}(x) + \left(n(2\lambda + n + 1) - \frac{m(2\lambda + m)}{1 - x^2} \right) \times P_{n,m}^{(\lambda)}(x) = 0, \tag{6}$$

which has the following solution:

$$\begin{aligned} P_{n,m}^{(\lambda)}(x) &= \frac{a_{n,m}(\lambda)}{(1 - x^2)^{\lambda + \frac{m}{2}}} \left(\frac{d}{dx} \right)^{n-m} ((1 - x^2)^{\lambda+n}) \\ &= \frac{a_{n,m}(\lambda)}{a_{n-m}(\lambda + m)} (1 - x^2)^{\frac{m}{2}} P_{n-m}^{(\lambda+m)}(x). \end{aligned} \tag{7}$$

The real constant $a_{n,m}(\lambda)$ is the normalization coefficient of the associated Gegenbauer function $P_{n,m}^{(\lambda)}(x)$. By choosing $m = 0$, it is clear that the differential equation of the associated Gegenbauer functions (6) is converted to the differential equation (1) for the Gegenbauer polynomials. If we choose the normalization coefficient $a_{n,m}(\lambda)$ as follows

$$a_{n,m}(\lambda) = a_{n-m}(\lambda + m), \tag{8}$$

then from Eqs. (5) and (7) one can obtain an inner product between the associated Gegenbauer functions:

$$\int_{-1}^{+1} P_{n,m}^{(\lambda)}(x)P_{n',m}^{(\lambda)}(x)(1-x^2)^{\lambda}\,dx = \delta_{nn'}h_{n-m}^2(\lambda+m) =: \delta_{nn'}h_{n,m}^2(\lambda) \quad n, n' \geq m. \tag{9}$$

The relation (9) indicates that the associated Gegenbauer functions with the different n's but the same m constitutes an orthogonal set. This formulation expresses that $a_{n,0}(\lambda) = a_n(\lambda)$, $h_{n,0}(\lambda) = h_n(\lambda)$, and $P_{n,0}^{(\lambda)}(x) = P_n^{(\lambda)}(x)$. Therefore, by determining norm of $P_n^{(\lambda)}(x)$ as $h_n(\lambda)$, norm of $P_{n,m}^{(\lambda)}(x)$ as $h_{n,m}(\lambda) = h_{n-m}(\lambda+m)$ are obtained.

2.3. Shape Invariance and Laddering Equations With Respect to *n*

Substituting the explicit forms of the raising and lowering operators

$$\begin{aligned} A_+(n;x) &= (1-x^2)\frac{d}{dx} - (2\lambda+n)x, \\ A_-(n;x) &= -(1-x^2)\frac{d}{dx} - nx, \end{aligned} \tag{10}$$

as well as the factorization spectrum

$$E(n,m) = (n-m)(2\lambda+n+m), \tag{11}$$

for a given m, we can factorize the associated differential Eq. (6) in form of shape invariance equations with respect to the parameter n:

$$\begin{aligned} A_+(n;x)A_-(n;x)P_{n,m}^{(\lambda)}(x) &= E(n,m)P_{n,m}^{(\lambda)}(x), \\ A_-(n;x)A_+(n;x)P_{n-1,m}^{(\lambda)}(x) &= E(n,m)P_{n-1,m}^{(\lambda)}(x). \end{aligned} \tag{12}$$

The shape invariance Eqs. (12) can be written in the form of raising and lowering equations with respect to the parameter n as below

$$A_+(n;x)P_{n-1,m}^{(\lambda)}(x) = \sqrt{E(n,m)}P_{n,m}^{(\lambda)}(x), \tag{13a}$$

$$A_-(n;x)P_{n,m}^{(\lambda)}(x) = \sqrt{E(n,m)}P_{n-1,m}^{(\lambda)}(x). \tag{13b}$$

Realizing shape invariance Eqs. (12) does not impose any condition on the normalization coefficients $a_{n,m}(\lambda)$, but realization of the raising and lowering relations (13) imposes a recurrence relation on these coefficients with respect to n:

$$a_{n,m}(\lambda) = \frac{2\lambda+n+m}{2(\lambda+n)}\frac{a_{n-1,m}(\lambda)}{\sqrt{E(n,m)}} \qquad n > m. \tag{14}$$

To get the recurrence relation (14), one has to substitute Eq. (7) in (13a) and then cancel the factor $(1-x^2)^{\frac{m}{2}}$ on both sides and also compare the coefficients of x^{n-m} in that relation. If we use a similar way for the relation (13b), and compare the coefficients of the highest power of x in both the sides, we will obtain a relation which is just an identity. By the application of the recurrence relation (14) in several times for a given m, we obtain

$$a_{n,m}(\lambda) = \left(\frac{1}{2}\right)^{n-m} \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+n+1)} \sqrt{\frac{\Gamma(2\lambda+n+m+1)}{\Gamma(n-m+1)\Gamma(2\lambda+2m+1)}}\, a_{m,m}(\lambda) \qquad n \geq m. \tag{15}$$

Exact determining functionallity $a_{n,m}(\lambda)$ of the parameters n, m, and λ will be done after the consideration of laddering equations with respect to the parameter m.

2.4. Shape Invariance and Laddering Equations With Respect to *m*

For a given n, the differential Eq. (6) can be also factorized with respect to m as the following shape invariance equations

$$\begin{aligned} A_+(m;x)A_-(m;x)P^{(\lambda)}_{n,m}(x) &= \varepsilon(n,m)P^{(\lambda)}_{n,m}(x), \\ A_-(m;x)A_+(m;x)P^{(\lambda)}_{n,m-1}(x) &= \varepsilon(n,m)P^{(\lambda)}_{n,m-1}(x), \end{aligned} \tag{16}$$

where

$$\begin{aligned} A_+(m;x) &= \sqrt{1-x^2}\,\frac{d}{dx} + \frac{(m-1)x}{\sqrt{1-x^2}}, \\ A_-(m;x) &= -\sqrt{1-x^2}\,\frac{d}{dx} + \frac{(2\lambda+m)x}{\sqrt{1-x^2}}, \end{aligned} \tag{17}$$

and

$$\varepsilon(n,m) = (n-m+1)(2\lambda+n+m). \tag{18}$$

In contrast to the previous case, the raising and lowering operators, i.e., $A_+(m;x)$ and $A_-(m;x)$ are Hermitian conjugate of each other respect to the inner product (9). The shape invariance Eqs. (16) can be written as the raising and lowering relations of the associated Gegenbauer functions:

$$A_+(m;x)P^{(\lambda)}_{n,m-1}(x) = \sqrt{\varepsilon(n,m)}\,P^{(\lambda)}_{n,m}(x), \tag{19a}$$

$$A_-(m;x)P^{(\lambda)}_{n,m}(x) = \sqrt{\varepsilon(n,m)}\,P^{(\lambda)}_{n,m-1}(x). \tag{19b}$$

In here also, realization of the shape invariance Eqs. (16) does not impose any condition on the $a_{n,m}(\lambda)$ but realization of the laddering Eqs. (19) imposes a recurrence relation with respect to m on the coefficients

$$a_{n,m}(\lambda) = -\sqrt{\varepsilon(n, m)}a_{n,m-1}(\lambda). \tag{20}$$

To prove (20), we must cancel the common factor $(1 - x^2)^{\frac{m}{2}}$ in every one of Eqs. (19a) and (19b), and then compare the coefficient of the highest power x, i.e., x^{n-m} and x^{n-m+1}, respectively, in both sides of them. For a given n, the recurrence relation (20) immediately gives

$$a_{n,m}(\lambda) = (-1)^{n-m}\sqrt{\frac{\Gamma(2\lambda + n + m + 1)}{\Gamma(n - m + 1)\Gamma(2\lambda + 2n + 1)}}a_{n,n}(\lambda) \quad m \le n. \tag{21}$$

Thus, to satisfy the raising and lowering relations of the parameter m as given in Eq. (19), we impose additional requirement in (21) on the coefficients $a_{n,m}(\lambda)$.

2.5. Simultaneous Realization of Laddering Equations With Respect to *n* and *m*

It is necessary that we notice the relations (15) and (21) in fact are imposed by the laddering relations (13) and (19). If we compare two different constraints (15) and (21), then we conclude that

$$a_{n,n}(\lambda) = \left(\frac{-1}{2}\right)^n \frac{\sqrt{\Gamma(2\lambda + 2n + 1)}}{\Gamma(\lambda + n + 1)}C(\lambda) \quad n = 0, 1, 2, \ldots, \tag{22}$$

in which the arbitrary real constant $C(\lambda)$ is independent of the parameters n and m. Note that the last equation is also valid for $n = m$. Thus, using every one of the relations (15) and (21), we can conclude

$$a_{n,m}(\lambda) = \frac{(-1)^m}{2^n\Gamma(\lambda + n + 1)}\sqrt{\frac{\Gamma(2\lambda + n + m + 1)}{\Gamma(n - m + 1)}}C(\lambda) \quad n \ge m. \tag{23}$$

To realize $P^{(\lambda)}_{n,0}(x) = P^{(\lambda)}_n(x)$, inserting the normalization coefficient (23) in (8), it becomes obvious that the constant $C(\lambda)$ must satisfy the following condition:

$$C(\lambda + m) = \left(\frac{-1}{2}\right)^m C(\lambda). \tag{24}$$

Also, using Eq. (23) for $m = 0$, in the relation (4) and then by imposing its result in equation $h_{n,m}(\lambda) = h_{n-m}(\lambda + m)$, we get the norm of the associated Gegenbauer functions $P^{(\lambda)}_{n,m}(x)$ as follows:

$$h_{n,m}(\lambda) = \sqrt{\frac{2^{2\lambda+1}}{2\lambda + 2n + 1}}C(\lambda) = h_n(\lambda). \tag{25}$$

Therefore, the necessity of the simultaneous realization of laddering equations with respect to n and m, i.e., (13) and (19), determines the normalization coefficient

$a_{n,m}(\lambda)$ and the norm $h_{n,m}(\lambda)$ in terms of the constant $C(\lambda)$ which is independent of the parameters n and m, as Eqs. (23) and (25). This means that the constant $C(\lambda)$ plays the role of a scale for the norm of the associated Gegenbauer functions $P_{n,m}^{(\lambda)}(x)$. Note that the norm $h_{n,m}(\lambda)$ is independent of m.

3. QUANTUM STATES SPLITTING ON THE ENERGY SPECTRAL OF A FREE PARTICLE ON THE MANIFOLDS AdS_2 AND S^2

The aim of this section is to split quantum states corresponding to the motion of a free particle on the manifolds of the anti-de Sitter AdS_2 and sphere S^2. It is shown that all these quantum states split to an infinite direct sum of infinite- and finite-dimensional Hilbert subspaces. So that, the Hilbert subspaces corresponding to AdS_2 and S^2 represent the dynamical symmetry groups $U(1, 1)$ and $U(2)$ with infinite- and finite-fold degeneracies, respectively. Using the change of variables $x = \tanh\theta$ and $x = -\cos\Theta$ and with the help of two new auxiliary variables $0 \leq \varphi < 2\pi$ and $0 \leq \phi < 2\pi$, one can write the following expressions for Eqs. (13) and (19)

$$J_+|n-1, m\rangle^{(AdS_2)} = \frac{h_n(\lambda)}{h_{n-1}(\lambda)}\sqrt{E(n, m)}|n, m\rangle^{(AdS_2)}, \tag{26a}$$

$$J_-|n, m\rangle^{(AdS_2)} = \frac{h_{n-1}(\lambda)}{h_n(\lambda)}\sqrt{E(n, m)}|n-1, m\rangle^{(AdS_2)}, \tag{26b}$$

and

$$L_+|n, m-1\rangle^{(S^2)} = \sqrt{\varepsilon(n, m)}|n, m\rangle^{(S^2)}, \tag{27a}$$

$$L_-|n, m\rangle^{(S^2)} = \sqrt{\varepsilon(n, m)}|n, m-1\rangle^{(S^2)}, \tag{27b}$$

respectively, in which the explicit forms of differential operators J_+ and J_- as well as L_+ and L_- are given by

$$\begin{aligned} J_+ &= e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\tanh\theta\frac{\partial}{\partial\varphi} - (1+2\lambda)\tanh\theta\right], \\ J_- &= e^{-i\varphi}\left[-\frac{\partial}{\partial\theta} + i\tanh\theta\frac{\partial}{\partial\varphi}\right], \end{aligned} \tag{28}$$

and

$$\begin{aligned} L_+ &= e^{i\phi}\left[\frac{\partial}{\partial\Theta} + i\cot\Theta\frac{\partial}{\partial\phi}\right], \\ L_- &= e^{-i\phi}\left[-\frac{\partial}{\partial\Theta} + i\cot\Theta\frac{\partial}{\partial\phi} - 2\lambda\cot\Theta\right]. \end{aligned} \tag{29}$$

The interval $-1 < x < +1$ transforms to the intervals $-\infty < \theta < +\infty$ and $0 < \Theta < \pi$ by the change of variables $x = \tanh\theta$ and $x = -\cos\Theta$, respectively.

We will see soon that θ and φ as well as Θ and ϕ are two appropriate pairs of variables for parameterization of a chart on AdS_2 and S^2, respectively. Therefore, in Eqs. (26) and (27) the kets $|n,m\rangle^{(AdS_2)}$ and $|n,m\rangle^{(s^2)}$ will be wave functions corresponding to the motion of a free particle on the manifolds AdS_2 and S^2:

$$|n,m\rangle^{(AdS_2)} = \frac{e^{in\varphi}}{\sqrt{2\pi}} \frac{P_{n,m}^{(\lambda)}(\tanh\theta)}{h_n(\lambda)}, \tag{30}$$

and

$$|n,m\rangle^{(S^2)} = \frac{e^{im\phi}}{\sqrt{2\pi}} \frac{P_{n,m}^{(\lambda)}(-\cos\Theta)}{h_n(\lambda)}. \tag{31}$$

Using the orthoghonallity relation (9), it is easy to prove that the kets $\{|n,m\rangle^{(AdS_2)}$ with $n \geq m\}$ for a given m, and the kets $\{|n,m\rangle^{(S^2)}$ with $n \geq m \geq 0\}$, constitute two separate orthonormal sets with respect to the inner products with measures $\cosh^{-2\lambda-2}\theta d\theta d\varphi$ and $\sin^{2\lambda+1}\Theta d\Theta d\phi$, respectively:

$$\begin{aligned} {}^{(AdS_2)}\langle n,m|n',m\rangle^{(AdS_2)} &:= \int_0^{2\pi}\int_{-\infty}^{+\infty} \left(\frac{e^{in\varphi}}{\sqrt{2\pi}} \frac{P_{n,m}^{(\lambda)}(\tanh\theta)}{h_n(\lambda)}\right)^* \\ &\quad \times \left(\frac{e^{in'\varphi}}{\sqrt{2\pi}} \frac{P_{n',m}^{(\lambda)}(\tanh\theta)}{h_{n'}(\lambda)}\right) \frac{d\theta\, d\varphi}{\cosh^{2\lambda+2}\theta} &(32a) \\ &= \delta_{nn'} \qquad n, n' \geq m &(32b) \end{aligned}$$

and

$$\begin{aligned} {}^{(S^2)}\langle n,m|n',m'\rangle^{(S^2)} &:= \int_0^{2\pi}\int_0^{\pi} \left(\frac{e^{im\phi}}{\sqrt{2\pi}} \frac{P_{n,m}^{(\lambda)}(-\cos\Theta)}{h_n(\lambda)}\right)^* \\ &\quad \times \left(\frac{e^{im'\phi}}{\sqrt{2\pi}} \frac{P_{n',m'}^{(\lambda)}(-\cos\Theta)}{h_{n'}(\lambda)}\right) \sin^{2\lambda+1}\Theta\, d\Theta\, d\phi &(33a) \\ &= \delta_{nn'}\delta_{mm'} \qquad n \geq m \quad \text{and} \quad n' \geq m'. &(33b) \end{aligned}$$

So we could construct the infinite- and $(n+1)$-dimensional Hilbert spaces equipped to inner products (32a) and (33a) corresponding to AdS_2 and S^2 with the orthonormal bases as follows, respectively:

$$\mathcal{H}_m := \text{span}\{|n,m\rangle^{(AdS_2)}\}_{n\geq m}, \tag{34}$$

and

$$\mathcal{H}_n := \text{span}\{|n,m\rangle^{(S^2)}\}_{m\leq n}. \tag{35}$$

It is clear that the operators J_+ and J_- with respect to the inner product (32a) are not Hermitian conjugation of each other while the operators L_+ and L_- with respect to (33a) are adjoint of each other. To complete the above mathematical structure, we define the following new operators

$$J_3 = -i\frac{\partial}{\partial\varphi}, \qquad I_J = 1, \tag{36}$$

and

$$L_3 = -i\frac{\partial}{\partial\phi}, \qquad I_L = 1, \tag{37}$$

which are represented by the Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_n$ as follows:

$$\begin{aligned} J_3|n, m\rangle^{(AdS_2)} &= n|n, m\rangle^{(AdS_2)}, \\ I_J|n, m\rangle^{(AdS_2)} &= |n, m\rangle^{(AdS_2)}, \end{aligned} \tag{38}$$

and

$$\begin{aligned} L_3|n, m\rangle^{(S^2)} &= m|n, m\rangle^{(S^2)}, \\ I_L|n, m\rangle^{(S^2)} &= |n, m\rangle^{(S^2)}. \end{aligned} \tag{39}$$

One can easily conclude that the operators J_+, J_-, J_3, and I_J, as well as L_+, L_-, L_3, and I_L, satisfy the commutation relations of Lie algebras $u(1, 1)$ and $u(2)$ as follows:

$$[J_+, J_-] = -2J_3 - 2\lambda - 1, \quad [J_3, J_\pm] = \pm J_\pm, \quad [\mathbf{J}, I_J] = 0 \tag{40}$$

and

$$[L_+, L_-] = 2L_3 + 2\lambda, \quad [L_3, L_\pm] = \pm L_\pm, \quad [\mathbf{J}, I_L] = 0. \tag{41}$$

Therefore, the generators J_+, J_-, J_3, and I_J, as well as L_+, L_-, L_3, and I_L of the Lie algebras $u(1, 1)$ and $u(2)$ are represented by the bases of infinite- and $(n + 1)$-dimensional Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_n$ as Eqs. (26, 38) and (27, 39), respectively. Now, it is clear that we deal with the discrete representations of the Lie algebra $u(1, 1)$ which are realized by $\mathcal{H}_m$. According to these equations, the Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_n$ are invariance under action of the generators of the Lie algebras $u(1, 1)$ and $u(2)$, respectively, i.e.

$$J_+, J_-, J_3, \quad I_J : \mathcal{H}_m \longrightarrow \mathcal{H}_m \qquad m = 0, 1, 2, \ldots, \tag{42}$$

and

$$L_+, L_-, \quad L_3, I_L : \mathcal{H}_n \longrightarrow \mathcal{H}_n \qquad n = 0, 1, 2, \ldots. \tag{43}$$

The Casimir operator of Lie algebras $u(1, 1)$ and $u(2)$ can be calculated as

$$\begin{aligned} H_J &= \frac{1}{2}\left[J_+J_- - J_3^2 - 2\lambda J_3 - S(\lambda)\right] \\ &= \frac{1}{2}\left[-\frac{\partial^2}{\partial\theta^2} + \frac{1}{\cosh^2\theta}\frac{\partial^2}{\partial\varphi^2} + 2\lambda\,\tanh\theta\frac{\partial}{\partial\theta} + i\frac{2\lambda+1}{\cosh^2\theta}\frac{\partial}{\partial\varphi} - S(\lambda)\right], \end{aligned} \tag{44}$$

and

$$\begin{aligned} H_L &= \frac{1}{2}\left[L_+L_- + L_3^2 + (2\lambda - 1)L_3 - S(\lambda)\right] \\ &= \frac{1}{2}\left[-\frac{\partial^2}{\partial\theta^2} - \frac{1}{\sin^2\Theta}\frac{\partial^2}{\partial\phi^2} - (2\lambda+1)\cot\Theta\frac{\partial}{\partial\Theta} - \frac{2i\lambda}{\sin^2\Theta}\frac{\partial}{\partial\phi} + 2\lambda - S(\lambda)\right], \end{aligned} \tag{45}$$

where $S(\lambda)$ is an arbitrary function in terms of λ. Now we are going to compare Casimir operators H_J and H_L with general form of Laplace–Beltrami operator $\mathcal{L} = -\frac{1}{2}D_j^A D^{jA} + V$ (Klinert, 1990) in terms of metric g_{ij}, gauge potential A_i, and the scalar potential V. The covariant derivative D_j^A is expressed in terms of gauge and Levi-Civita connections A_j and ∇_j as $D_j^A = \nabla_j - iA_j$. The index j takes the values θ and φ, as well as Θ and ϕ for the Lie algebras $u(1,1)$ and $u(2)$, respectively. If we use the explicit form of the Casimir operators H_J and H_L instead of $\mathcal{L}$ and compare the coefficients of second-order partial derivatives, then we obtain the induced metrics $g_{ij}^{(AdS_2)}$ and $g_{ij}^{(S^2)}$ for two-dimensional ordinary Riemannian manifolds which have been parameterized by the variables θ and φ as well as Θ and ϕ:

$$g_{ij}^{(AdS_2)} = \begin{pmatrix} 1 & 0 \\ 0 & -\cosh^2\theta \end{pmatrix}, \quad g_{ij}^{(S^2)} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\Theta \end{pmatrix}. \tag{46}$$

The nonzero components Christoffel symbols and Ricci tensor corresponding to the metrics (46) are calculated as

$$\begin{aligned} &\Gamma^{\theta}_{\varphi\varphi} = \frac{1}{2}\sinh 2\theta \quad \Gamma^{\varphi}_{\theta\varphi} = \tanh\theta \quad R_{\theta\theta} = -1 \quad R_{\varphi\varphi} = \cosh^2\theta, \\ &\Gamma^{\Theta}_{\phi\phi} = \frac{-1}{2}\sin 2\Theta \quad \Gamma^{\phi}_{\Theta\phi} = \cot\Theta \quad R_{\Theta\Theta} = 1 \quad R_{\phi\phi} = \sin^2\Theta. \end{aligned} \tag{47}$$

So Ricci scalar curvature $R = g^{ij}R_{ij}$ of the manifolds which are described by the metrics (46) are

$$R^{(AdS_2)} = -2, \qquad R^{(S^2)} = 2. \tag{48}$$

Therefore, the variables θ and φ, as well as Θ and ϕ via the metrics $g_{ij}^{(AdS_2)}$ and $g_{ij}^{(S^2)}$ given in (46), describe the Riemannian manifolds AdS_2 and S^2. It is clear that the coordinate systems θ and φ as well as Θ and ϕ describe the appropriate charts for parameterizing one of the connected components AdS_2 and S^2 as local, respectively. Comparing the coefficients of the first-order partial derivatives in the Laplace–Beltrami and Casimir operators, we get

$$\begin{aligned} &A_\theta = \frac{-i}{2}(2\lambda+1)\tanh\theta \qquad A_\varphi = -\frac{2\lambda+1}{2}, \\ &A_\Theta = \frac{-i}{2}\cot\Theta \qquad A_\phi = -\lambda. \end{aligned} \tag{49}$$

Hence two forms of magnetic fields corresponding to the components of gauge potentials (49) vanish

$$\mathbf{B} = \frac{dA_\theta}{d\theta}\, d\theta \wedge d\theta + \frac{dA_\varphi}{d\theta}\, d\theta \wedge d\varphi = 0. \tag{50}$$

Also, by comparing the remainder of terms in the Laplace–Beltrami and Casimir operators, the scalar potential V can be obtained:

$$\begin{aligned} V^{(AdS_2)} &= \frac{-1}{2}\left[\frac{1}{4} - \lambda^2 + S(\lambda)\right] \Longrightarrow \mathbf{E} = -\frac{dV}{d\theta}\,\mathbf{e}_\theta = 0, \\ V^{(S^2)} &= \frac{-1}{2}[\lambda(\lambda-1) + S(\lambda)] \Longrightarrow \mathbf{E} = -\frac{dV}{d\Theta}\,\mathbf{e}_\Theta = 0. \end{aligned} \tag{51}$$

The Casimir operators H_J and H_L satisfy the following eigenvalue equations:

$$H_J|n, m\rangle^{(AdS_2)} = E_J(m)|n, m\rangle^{(AdS_2)} \quad E_J(m) = \frac{-1}{2}[m(m+2\lambda) + S(\lambda)] \;\; n \geq m, \tag{52}$$

$$H_L|n, m\rangle^{(S^2)} = E_J(n)|n, m\rangle^{(S^2)} \quad E_L(n) = \frac{1}{2}[(n+1)(n+2\lambda) - S(\lambda)] \;\; m \leq n. \tag{53}$$

So the Casimir operators H_J and H_L are Hamiltonians corresponding to the motion of a free particle on AdS_2 and S^2, so that all bases of the Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_n$ are eigenstates of them with the same eigenvalues $E_J(m)$ and $E_L(n)$, respectively. These equations mean that the representation of the Casimir operators H_J and H_L by the Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_n$ have the dynamical symmetry groups $U(1, 1)$ and $U(2)$ with the infinite- and $(n+1)$-fold degeneracies, respectively,

$${}^{(AdS_2)}\langle n', m|H_J|n, m\rangle^{(AdS_2)} = E_J(m)\delta_{n'n} \quad n, n' \geq m, \tag{54}$$

$${}^{(S^2)}\langle n, m'|H_L|n, m\rangle^{(S^2)} = E_L(n)\delta_{m'm} \quad m, m' \leq n. \tag{55}$$

Now we can introduce the Hilbert spaces $\mathcal{H}^{(AdS_2)}$ and $\mathcal{H}^{(S^2)}$ as infinite direct sums of the infinite- and $(n+1)$-dimensional Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_n$, respectively:

$$\mathcal{H}^{(AdS_2)} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m = \text{span}\left\{|n, m\rangle^{(AdS_2)}\right\}_{n \geq m \geq 0}, \tag{56}$$

$$\mathcal{H}^{(S^2)} = \bigoplus_{m=0}^{\infty} \mathcal{H}_n = \text{span}\left\{|n, m\rangle^{(S^2)}\right\}_{n \geq m \geq 0}. \tag{57}$$

The relations (56) and (57) represent quantum states splitting on the same energy spectral corresponding to the motion of a free particle on the manifolds AdS_2 and S^2 into the representation spaces of the Lie algebras $u(1, 1)$ and $u(2)$, respectively.

Now we try to find first-order differential maps that one of them maps the Hilbert subspace $\mathcal{H}_{m-1}$ to $\mathcal{H}_m$ (and $\mathcal{H}_{n-1}$ to $\mathcal{H}_n$) and other one maps the Hilbert subspace $\mathcal{H}_m$ to $\mathcal{H}_{m-1}$ (and $\mathcal{H}_n$ to $\mathcal{H}_{n-1}$). Writing the operators $A_+(m; x)$, $A_-(m; x)$, (and $A_+(n; x)$, $A_-(n; x)$) in terms of the variable θ (and Θ) as

$$\begin{aligned} A_+(m; \tanh\theta) &= \cosh\theta \frac{\partial}{\partial\theta} + (m-1)\sinh\theta, \\ A_-(m; \tanh\theta) &= -\cosh\theta \frac{\partial}{\partial\theta} + (2\lambda + m)\sinh\theta, \end{aligned} \tag{58}$$

and

$$\begin{aligned} A_+(n; -\cos\Theta) &= \sin\Theta \frac{\partial}{\partial\Theta} + (2\lambda + n)\cos\Theta, \\ A_-(n; -\cos\Theta) &= -\sin\Theta \frac{\partial}{\partial\Theta} + n\cos\Theta, \end{aligned} \tag{59}$$

we can conclude that they map Hilbert subspaces $\mathcal{H}_m$ (and $\mathcal{H}_n$) to each other as follows:

$$\begin{aligned} A_+(m; \tanh\theta) &: \mathcal{H}_{m-1} \longrightarrow \mathcal{H}_m, \\ A_-(m; \tanh\theta) &: \mathcal{H}_m \longrightarrow \mathcal{H}_{m-1}, \end{aligned} \tag{60}$$

and

$$\begin{aligned} A_+(n; -\cos\Theta) &: \mathcal{H}_{n-1} \longrightarrow \mathcal{H}_n, \\ A_-(n; -\cos\Theta) &: \mathcal{H}_n \longrightarrow \mathcal{H}_{n-1}. \end{aligned} \tag{61}$$

Using Eqs. (19) and (13), one can obtain the following rules for these mappings:

$$\begin{aligned} A_+(m; \tanh\theta)|n, m-1\rangle^{(AdS_2)} &= \sqrt{\varepsilon(n, m)}|n, m\rangle^{(AdS_2)}, \\ A_-(m; \tanh\theta)|n, m\rangle^{(AdS_2)} &= \sqrt{\varepsilon(n, m)}|n, m-1\rangle^{(AdS_2)}, \end{aligned} \tag{62}$$

and

$$\begin{aligned} A_+(n; -\cos\Theta)|n-1, m\rangle^{(S^2)} &= \frac{h_n(\lambda)}{h_{n-1}(\lambda)}\sqrt{E(n, m)}|n, m\rangle^{(S^2)}, \\ A_-(n; -\cos\Theta)|n, m\rangle^{(S^2)} &= \frac{h_{n-1}(\lambda)}{h_n(\lambda)}\sqrt{E(n, m)}|n-1, m\rangle^{(S^2)}. \end{aligned} \tag{63}$$

The differential operators $A_+(m; \tanh\theta)$, $A_-(m; \tanh\theta)$, and $(A_+(n; -\cos\Theta)$, $A_-(n; -\cos\Theta))$ map quantum states corresponding to the motion of a free particle on AdS_2 (and S^2) from a Hilbert subspace to an another Hilbert subspace so that their energies change. While according to equations (26), (52), and (27, 53) the operators J_+, J_-, (and L_+, L_-) transform these quantum states inside of a Hilbert subspace so that their energies remain without change.

Equations (26b) and (27a) show that the lowest and highest states $|m, m\rangle^{(AdS_2)}$ and $|n, n\rangle^{(S^2)}$ belonging to $\mathcal{H}_m$ and $\mathcal{H}_n$ satisfy the following first-order differential equations

$$J_-|m, m\rangle^{(AdS_2)} = 0, \tag{64}$$

and

$$L_+|n, n\rangle^{(S^2)} = 0, \tag{65}$$

with the following solutions

$$|m, m\rangle^{(AdS_2)} = \frac{a_{m,m}(\lambda)}{\sqrt{2\pi}h_m(\lambda)} \frac{e^{im\varphi}}{\cosh^m \theta}, \tag{66}$$

and

$$|n, n\rangle^{(S^2)} = \frac{a_{n,n}(\lambda)}{\sqrt{2\pi}h_n(\lambda)} e^{in\phi} \sin^n \Theta. \tag{67}$$

From the analytic solutions (30) and (31) we can also get the the lowest and the highest states $|m, m\rangle^{(AdS_2)}$ and $|n, n\rangle^{(S^2)}$ as (66) and (67). So, with the help of (26a) and (27b), one can obtain the arbitrary quantum states in the framework of the algebraic manners as

$$|n, m\rangle^{(AdS_2)} = \frac{h_m(\lambda)}{h_n(\lambda)} \frac{J_+^{n-m}|m, m\rangle^{(AdS_2)}}{\sqrt{E(n, m)E(n-1, m)\ldots E(m+1, m)}} \quad n \geq m+1, \tag{68}$$

and

$$|n, m\rangle^{(S^2)} = \frac{L_-^{n-m}|n, n\rangle^{(S^2)}}{\sqrt{\varepsilon(n, m+1)\varepsilon(n, m+2)\ldots\varepsilon(n, n)}} \quad m \leq n-1. \tag{69}$$

Those make sense that all bases of Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_n$ are generated by successive action of the operators J_+ and L_- on the lowest and highest states, respectively.

4. EMBEDDING OF LIE ALGEBRA $gl(2, C)$ INTO THE PARASUPERSYMMETRY ALGEBRA

The general Lie algebra $gl(2, C)$ with the rank 1 is the complexification of the real Lie algebras $h_4, u(2), u(1, 1), iso(2) \oplus u(1), su(2) \oplus u(1)$ and $su(1, 1) \oplus u(1)$. In this section we use the generators of the Lie algebra $gl(2, C)$ in the Cartan bases $\{K_+, K_-, K_3, I_K\}$ with the following commutation relations

$$[K_+, K_-] = uK_3 + vI_K, \tag{70a}$$

$$[K_3, K_\pm] = \pm K_\pm, \tag{70b}$$

$$[\boldsymbol{K}, I_K] = 0, \tag{70c}$$

where u and v are the real structure constants. It is clear that the commutation relations (70) in the cases $u = 0$, $v \neq 0$, and $u = 0$, $v = 0$ are the Heisenberg Lie algebra h_4 and Lie algebra $iso(2) \oplus u(1)$, respectively. Also in cases $u > 0, v \neq 0$, and $u < 0, v \neq 0$ they describe the commutation relations of Lie algebras $u(2)$ and $u(1, 1)$, respectively. In the recent two cases if v is equal to zero then we will deal with Lie algebras $su(2) \oplus u(1)$ and $su(1, 1) \oplus u(1)$, respectively. We will also use representation space of Lie algebra $gl(2, C)$ combined from basis kets $|l\rangle$ which are realizing the representation of Lie algebra as follows:

$$\begin{aligned} K_+|l-1\rangle &= \frac{h_l}{h_{l-1}}\sqrt{E_{gl}(l)}|l\rangle, \\ K_-|l\rangle &= \frac{h_{l-1}}{h_l}\sqrt{E_{gl}(l)}|l-1\rangle, \\ K_3|l\rangle &= l|l\rangle, \\ I_K|l\rangle &= |l\rangle. \end{aligned} \tag{71}$$

To construct the correspondence between the parasupersymmetry algebra of arbitrary order p and Lie algebra $gl(2, C)$, we do not use any information about that does l have up or down limitations. It is clear that for realizing the representation (71) by kets $|l\rangle$, commutation relations (70b) and (70c) are identically satisfied, but the commutation relation (70a) imposes the following constraint on the representation spectrum $E_{gl}(l)$ and the structure constants u and v:

$$E_{gl}(l) - E_{gl}(l+1) = lu + v. \tag{72}$$

In order to realize representation of Lie algebra $gl(2, C)$ by kets $|l\rangle$ in both cases bounded of down, $l \geq r$, and bounded of up, $l \leq r$ (r is an arbitrary positive integer), using the recurrence relation (72), one can separately obtain the following same results:

$$E_{gl}(l) = E_{gl}(r) - \frac{1}{2}(l-r)[(l+r-1)u + 2v]. \tag{73}$$

We want to show that there is a deep connection between Lie algebra $gl(2, C)$ with rank 1 and the Khare–Rubakov–Spiridonov nonunitary parasupersymmetry algebra of arbitrary order, and it is the fact that Lie algebra $gl(2, C)$ is able to represent parasupersymmetry algebra.

The Khare–Rubakov–Spiridonov nonunitary parasupersymmetry algebra of arbitrary order p with two parafermionic generators Q_1 and Q_2 also one bosonic generator H is known with the following relations:

$$Q_1^p Q_2 + Q_1^{p-1} Q_2 Q_1 + \cdots + Q_1 Q_2 Q_1^{p-1} + Q_2 Q_1^p = 2p Q_1^{p-1} H, \tag{74a}$$

$$Q_2^p Q_1 + Q_2^{p-1} Q_1 Q_2 + \cdots + Q_2 Q_1 Q_2^{p-1} + Q_1 Q_2^p = 2p Q_2^{p-1} H, \tag{74b}$$

$$Q_1^{p+1} = Q_2^{p+1} = 0, \tag{74c}$$

$$[H, Q_1] = [H, Q_2] = 0. \tag{74d}$$

Nonunitarity means that parafermions Q_1 and Q_2 are not Hermitian conjugate of each other. Therefore, it is not necessary that the bosonic operator H should be Hermitian. To realize the algebraic relations (74), let us define firstly the operators Q_1, Q_2, and H as matrices $(p+1)\times(p+1)$ with the following elements:

$$(Q_1)_{ss'} := K_- \delta_{s+1 s'}, \tag{75a}$$

$$(Q_2)_{ss'} := K_+ \delta_{s s'+1}, \tag{75b}$$

$$(H)_{ss'} := H_s \delta_{ss'}, \quad s, s' = 1, 2, \ldots, p+1. \tag{75c}$$

The definitions (75a) and (75b) satisfy automatically Eqs. (74c). We will find the operators H_s in the interests of satisfying Eqs. (74a,b,d). The multilinear Eqs. (74a,b) and also commutation relations (74d) for the generators defined by (75) lead us to the following equations:

$$K_-^p K_+ + K_-^{p-1} K_+ K_- + \cdots + K_- K_+ K_-^{p-1} = 2p K_-^{p-1} H_p, \tag{76a}$$

$$K_-^{p-1} K_+ K_- + \cdots + K_- K_+ K_-^{p-1} + K_+ K_-^p = 2p K_-^{p-1} H_{p+1}, \tag{76b}$$

$$K_+^{p-1} K_- K_+ + \cdots + K_+ K_- K_+^{p-1} + K_- K_+^p = 2p K_+^{p-1} H_1, \tag{76c}$$

$$K_+^p K_- + K_+^{p-1} K_- K_+ + \cdots + K_+ K_- K_+^{p-1} = 2p K_+^{p-1} H_2, \tag{76d}$$

and

$$\begin{aligned} H_s K_- &= K_- H_{s+1}, \\ H_{s+1} K_+ &= K_+ H_s \qquad s = 1, 2, \ldots, p. \end{aligned} \tag{77}$$

To satisfy Eqs. (76) and (77), we suggest the dependence of operators $H_1, H_2, \ldots,$ H_{p+1} to the generators of Lie algebra $gl(2, C)$ as

$$\begin{aligned} H_s &= \frac{1}{2}(K_- K_+ + c_s K_3 + d_s) \qquad s = 1, 2, \ldots, p, \\ H_{p+1} &= \frac{1}{2}(K_+ K_- + c_{p+1} K_3 + d_{p+1}). \end{aligned} \tag{78}$$

Now the coefficients c_s and d_s, $s = 1, 2, \ldots, p+1$, in terms of structure constants u and v and also p must be determined so that the relations (76) and (77) are satisfied by the Hamiltonians H_s.

The relations (77) impose the following recurrence relations upon on the coefficients c_s and d_s:

$$\begin{aligned} c_{s+1} &= c_s + (1 - \delta_{sp}) u, \\ d_{s+1} &= d_s - c_s + (1 - \delta_{sp}) v \qquad s = 1, 2, \ldots, p. \end{aligned} \tag{79}$$

Using the recurrence relations (79) for several times, we get

$$c_s = (s - 1 - \delta_{sp+1})u + c_1,$$
$$d_s = \frac{-1}{2}(s-1)(s-2)u + (s - 1 - \delta_{sp+1})v - (s-1)c_1 + d_1$$
$$s = 1, 2, \ldots, p+1. \qquad (80)$$

The relations (80) show that if c_1 and d_1 become determined then all the coefficients c_s and d_s will be found. Since H_1 has been used in the right-hand side of relation (76c), we may achieve c_1 and d_1 directly from this relation. It is sufficient that the commutation relations of Lie algebra $gl(2, C)$ given in (70) are repeatedly used:

$$c_1 = -\frac{1}{2}(p-1)u$$
$$d_1 = -\frac{1}{2}(p-1)\left(\frac{1}{3}(p+1)u + v\right). \qquad (81)$$

Substituting the results (81) into (80), we obtain

$$c_s = -\frac{1}{2}(p - 2s + 1 + 2\delta_{sp+1})u$$
$$d_s = -\frac{1}{6}(3s^2 - 3sp + p^2 + 3p - 6s + 2)u - \frac{1}{2}(p - 2s + 1 + 2\delta_{sp+1})v. \quad (82)$$

Finally, substituting the relations (82) into (78), we can find the components of bosonic operator as follows:

$$H_s = \frac{1}{2}\left[K_-K_+ - \frac{1}{2}(p-2s+1)uK_3 - \frac{1}{6}(3s^2 - 3sp + p^2 + 3p - 6s + 2)u \right.$$
$$\left. -\frac{1}{2}(p-2s+1)v\right] \qquad s = 1, 2, \ldots, p+1. \qquad (83)$$

Using Eqs. (71) and (73), we can easily show that for given l and r, the operators H_s satisfy the following eigenvalue equations

$$H_s|l+s\rangle = E_{gl}(l, r, p)|l+s\rangle \qquad s = 1, 2, \ldots, p+1, \qquad (84)$$

where

$$E_{gl}(l, r, p) = \frac{1}{4}\left[2E_{gl}(r) - \left(l^2 - r^2 + 2l + r + lp + \frac{p^2}{3} + p + \frac{2}{3}\right)u \right.$$
$$\left. -(2l - 2r + p + 3)v\right]. \qquad (85)$$

Note that if the representation kets $|l\rangle$ be as $l \geq r$, then we have automatically $l + s \geq r$, whereas if the representation kets $|l\rangle$ be as $l \leq r$, then we must choose kets $|l\rangle$ so that $l + p + 1 \leq r$. The recent result implies that the eigenvalue of operators H_s are independent of s and the importance of this more in representation of nonunitary parasupersymmery algebra of arbitrary order will be appeared. Now using Eq. (83), we can substitute the explicit form of operators H_p, H_{p+1}, and H_2 in Eqs. (76a), (76b), and (76d), respectively, and with the long calculation, show that these equations are also satisfied. To complete this part of our discussion, the question "How is represented the nonunitary parasupersymmetry algebra of arbitrary order?" should be answered. For given l and r with $l \geq r$ or $l + p + 1 \leq r$, parastate of nonunitary parasupersymmetry algebra of arbitrary order p is defined as follows:

$$|l\rangle_{\text{pss}} = \begin{pmatrix} |l+1\rangle \\ |l+2\rangle \\ |l+3\rangle \\ \vdots \\ |l+p+1\rangle \end{pmatrix}. \tag{86}$$

By using Eqs. (75c) and (84), it becomes obvious that the parastates $|l\rangle_{\text{pss}}$ represent the bosonic operator H as below

$$H|l\rangle_{\text{pss}} = E_{gl}(l, r, p)|l\rangle_{\text{pss}}. \tag{87}$$

Defining the matrices $(p + 1) \times (p + 1)$, q_i, $i = 1, 2, \ldots, p + 1$, as

$$(q_i)_{jk} = \delta_{ij}\delta_{jk}, \tag{88}$$

we obtain the following eigenvalue equations for parafermionic operators Q_1 and Q_2

$$\begin{aligned} Q_1|l\rangle_{\text{pss}} &= \left(\sum_{i=1}^{p} \frac{h_{l+1}}{h_{l+i+1}} \sqrt{E_{gl}(l+i+1)}q_i\right)|l\rangle_{\text{pss}}, \\ Q_2|l\rangle_{\text{pss}} &= \left(\sum_{i=1}^{p} \frac{h_{l+i+1}}{h_{l+i}} \sqrt{E(l+i+1)}q_{i+1}\right)|l\rangle_{\text{pss}}. \end{aligned} \tag{89}$$

It is evident that the algebraic relations (74), for the parafermionic generators Q_1 and Q_2 and also bosonic generator H given in Eqs. (75) and (83), are also satisfied on the parastate $|l\rangle_{\text{pss}}$. It should be noticed that the technique used above does not need to make generators of parasupersymmetry algebra by the generator I_k. This means that if $v = 0$ it depends on weather $u > 0$ or $u < 0$, automatically the generators of Lie algebras $su(2)$ or $su(1, 1)$ for realizing the representation of parasupersymmetry algebra are used. All things show that the nonunitary parasupersymmetry algebra of arbitrary order p is represented by each of real forms of Lie algebra $gl(2, C)$, including Lie algebra $u(2)$ and its subalgebra $su(2)$, Lie algebra

$u(1, 1)$ and its subalgebra $su(1, 1)$, Lie algebra $iso(2) \oplus u(1)$, and its subalgebra $iso(2)$, and also, contracted form $gl(2, C)$, i.e., h_4.

If, instead of $gl(2, C)$, we consider Lie algebras $u(1, 1)$ and $u(2)$ as (40) and (41) with representations (26), (38), and (27), (39), then our discussions are assigned to the motion of a free particle on AdS_2 and S^2. We choose the generators $\{K_+, K_-, K_3, I_K\}$ as $\{J_+, J_-, J_3, I_J\}$ and $\{L_+, L_-, L_3, I_L\}$ for them; $l = n, r = m, n \geq m$, and $l = m, r = n, m \leq n$; $u = -2, v = -2\lambda - 1$, and $u = 2, v = 2\lambda$; $E_{gl}(l) = E(n, m)$, $E_{gl}(r) = E(m, m) = 0$ and $E_{gl}(l) = \varepsilon(n, m) = E_{gl}(r) = \varepsilon(n, n) = 2(\lambda + n)$; $h_1 = h_n(\lambda)$ and $h_l = 1$; respectively. Therefore, the Hilbert subspaces $\mathcal{H}_m$ and $\mathcal{H}_n$, i.e., quantum states corresponding to the motion of a free particle on AdS_2 and S^2 with the dynamical symmetry groups $U(1, 1)$ and $U(2)$, can be used for realizing representation of nonunitary parasupersymmetry algebra of arbitrary orders p and $p \leq n$, respectively.

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